

Classical dynamical r -matrices for Calogero-Moser systems and their generalizations

J. Avan

L.P.T.H.E. Université Paris VI (CNRS UA 280), Box 126, Tour 16, 1^{er} étage, 4 place Jussieu, 75252 Paris Cedex 05, France

1 Introduction

The construction and study of classical (and quantum) dynamical r -matrices is currently undergoing extensive development. Various examples of such objects were recently discussed, for instance in [1, 2, 3]. However at this time one lacks a general classifying scheme such as exists in the case of constant classical r -matrices thanks to Belavin and Drin'feld [4]. A partial classification scheme has very recently been proposed [5] for dynamical r -matrices obeying the particular version of the dynamical Yang-Baxter equation [6] corresponding to Calogero-Moser models [7].

Consideration of such structures is thus particularly relevant to the study of Calogero-Moser [8, 9] and relativistic Ruijsenaars-Schneider models [10, 11] where they appear systematically. Their occurrence and the particular form they assume, which we shall give in detail in this lecture, are due to the common nature of these models as hamiltonian reductions of free or harmonic motions on particular symplectic manifolds: cotangent bundle of Lie algebras or Lie groups for rational or trigonometric Calogero-Moser models [12]; double of Lie groups [13] or cotangent bundle to centrally extended loop group [14] for Ruijsenaars-Schneider models. The most general elliptic potentials are in turn associated to loop groups over elliptic curves [14, 13] and are crucial in understanding the algebraic resolution of these models [15, 16].

This hamiltonian reduction procedure is also an important tool in the explicit construction of classical r -matrices for BC_n -type systems, where the direct resolution of the intricate r -matrices equations, such as was done in [17], becomes untractable. We shall exemplify such a construction in the case of trigonometric Calogero-Moser models [18].

The plan of this lecture runs as follows. We first recall the essential results of classical r -matrix theory and introduce the notations to be used throughout it. We then describe the construction of classical r -matrices for trigonometric Calogero-Moser models using the Hamiltonian reduction procedure. This gives us a general formula valid for all non-exceptional Lie groups.

We finally give a systematic overall picture of the classical r -matrix structure of this type which were obtained by various authors for:

- a) elliptic, trigonometric and rational A_n Calogero-Moser models [17]
- b) elliptic, trigonometric and rational A_n spin Calogero-Moser models [19]
- c) elliptic, trigonometric and rational A_n Ruijsenaars-Schneider models. [20, 21, 22].

The case of spin Ruijsenaars-Schneider systems [11] was recently investigated in the rational case [23] but the general Hamiltonian structure still escapes understanding at this time.

To be complete we must indicate that two alternative approaches were recently described. One, using the same Lax matrices but leading to dynamical r -matrices of a different type (to be commented upon later) was developed in [24] (see contribution by Pr. Braden). The other one uses conjugated Lax matrices which allow to eliminate the dynamical dependence in the r -matrix at the cost of introducing a more complicated Lax operator. It was developed directly in the quantum case using the formalism of intertwining vectors [25] (see contribution by Pr. Hasegawa).

2 Preliminaries

First of all we need to recall four essential features of the classical r -matrix formalism (see [26] for a textbook presentation of the hamiltonian theory of classical integrable systems.).

We consider a generic dynamical system described by a set of coordinates $\{x_i\}$ and momenta $\{p_i\}$, $i = 1 \cdots n$; a Poisson structure $\{\}$ and a Hamiltonian $h(x_i, p_j)$.

1- Liouville theorem

The existence of n algebraically independent, globally defined, Poisson commuting quantities such that the hamiltonian h belongs to the ring generated by this set, guarantees the existence of a canonical transformation $(x_i, p_j) \rightarrow (I_i, \theta_j)$ linearizing the equations of motion [27]. Further assumptions on the topological structure of the phase space allow more precise statements on the geometrical interpretation of the transformed variables known as action-angle variables [28].

2- Lax pair formulation

The Lax pair formulation of a dynamical system is the giving of two elements of a Lie algebra \mathcal{G} , $L(x, p)$ and $M(x, p)$ such that the equations of motion for x_i, p_i are equivalent to the isospectral evolution (Lax equation) [29]

$$\frac{dL}{dt} = [L, M] \quad (1)$$

It follows that the adjoint-invariant quantities $Tr L^n$ are time-independent. In order to implement Liouville theorem onto this set of possible action variables we need them to be Poisson-commuting. This is ensured by the classical r matrix structure.

3- The r -matrix structure

Defining the decomposition of the Lax operator on a basis $\{t_a\}$ of the Lie algebra \mathcal{G} as $L \equiv \sum_a l^a t_a$, the Poisson commutation of the ad-invariant $Tr L^n$ is equivalent to the existence of an object $r_{12}(x, p) \in \mathcal{G}_1 \otimes \mathcal{G}_2$ hereafter known as a classical r -matrix [30, 31, 32], such that:

$$\{L_1, L_2\} \equiv \sum_{a,b} \{l^a, l^b\} t_a \otimes t_b = [r_{12}, L_1] - [r_{21}, L_2] \quad (2)$$

It must immediately be remarked that such an object is by no means unique. Moreover there is no one-to-one correspondance between a given dynamical system and the Lie algebra in which its Lax representation is defined; a same dynamical system may have several Lax representations and several r -matrix structures.

4- The classical Yang-Baxter equation

The Poisson bracket structure (2) obeys a Jacobi identity which implies an algebraic constraint for the r -matrix. Since r depends a priori on the dynamical variables this constraint takes a complicated form :

$$[L_1, [r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{32}, r_{13}] + \{L_2, r_{13}\} - \{L_3, r_{12}\}] + \text{cycl. perm.} = 0 \quad (3)$$

Relevant particular cases of this very general identity are obtained when:

a) r is independant of x, q . One is then lead to the general non-dynamical Yang Baxter equation [30, 31, 32, 33]:

$$[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{32}, r_{13}] = 0 \quad (4)$$

b) If furthermore r is antisymmetric under permutation of the two copies of the algebra \mathcal{G} one obtains the better known and much studied [30, 4] form:

$$[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0 \quad (5)$$

c) If on the contrary r is dynamical, the supplementary terms in (3) may take a completely algebraic form due to the specific structure of the Lax operator. For instance the Calogero-Moser models lead to a self-contained algebraic equation for a classical r matrix depending only on position-type canonical variables x_ν :

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] - \sum_\nu h_\nu^{(1)} \frac{\partial}{\partial x_\nu} r_{23} + \sum_\nu h_\nu^{(2)} \frac{\partial}{\partial x_\nu} r_{13} - \sum_\nu h_\nu^{(3)} \frac{\partial}{\partial x_\nu} r_{12} = 0 \quad (6)$$

where the set $\{h_\nu^{(i)}\}$ is a choice of basis for the Cartan algebra of \mathcal{G} acting on the representation space i . This equation was first derived in [6] and a classification scheme of its solutions was proposed in [5], closely connected to the general algebraic scheme in [4].

Note by the way that there exists canonical examples of dynamical r matrices, obtained by using the well-known higher Poisson bracket construction for any integrable dynamical system [34, 35] starting from a constant r -matrix. In particular the quadratic Sklyanin bracket, where the Poisson structure of a Lax matrix becomes $\{L_1, L_2\} = [r_{12}, L_1 \otimes L_2]$, may also be described as a linear structure with a dynamical r matrix $R_{12} \equiv r_{12} L_2$. We shall see on the example of the Ruijsenaars-Schneider system that dynamical “linear” r -matrix structures may also give rise to dynamical “quadratic” structures. The *initial* dynamical r -matrices themselves, however, are themselves *not* of the Sklyanin type.

3 Hamiltonian reduction and r -matrices

3.1 General hamiltonian reduction

We begin by recalling some well-known facts concerning the Hamiltonian reduction of dynamical systems whose phase space is a cotangent bundle [36]. Let M be a manifold and $N = T^*M$ its cotangent bundle. N is equipped with the canonical 1-form α whose value at the point $p \in T^*M$ is π^*p where π is the projection of N on M . If a Lie group G acts on M , each element $X \in \mathcal{G}$ (the Lie algebra of G) generates a vector field on M that we shall

denote $X.m$ at the point $m \in M$. It lifts to a vector field on N leaving α invariant. We shall also denote $X.p \in T_p(N)$ the value at $p \in N$ of this vector field, so that the Lie derivative $\mathcal{L}_{X.p} \alpha$ of the canonical 1-form vanishes.

N is a symplectic manifold equipped with the canonical 2-form $\omega = -d\alpha$. To any function (or Hamiltonian) H on N we associate a vector field X_H such that $dH = i_{X_H} \omega$ and conversely since ω is non-degenerate.

The Hamiltonian associated to the above vector field $X.p$, $X \in \mathcal{G}$ reads:

$$H_X(p) = i_{X.p} \alpha = \alpha(X.p) \quad (7)$$

For any two functions F, G on N one defines the Poisson bracket $\{F, G\}$ as a function on N by:

$$\{F, G\} = \omega(X_F, X_G) \quad (8)$$

The Poisson bracket of the Hamiltonians associated to the group action has a simple expression. In fact the group action is Poissonnian, i.e.:

$$\{H_X, H_Y\} = H_{[X, Y]} \quad (9)$$

Obviously the application $X \in \mathcal{G} \rightarrow H_X(p)$, any for any $p \in N$, is a linear map from \mathcal{G} to the scalars and so defines an element $\mathcal{P}(p)$ of \mathcal{G}^* which is called the momentum at $p \in N$.

One then restricts oneself to the submanifold N_μ of N with fixed momentum μ i.e. such that: $N_\mu = \mathcal{P}^{-1}(\mu)$

Due to equation (7) and the invariance of α the action of the group G on N is transformed by \mathcal{P} into the coadjoint action of G on \mathcal{G}^*

$$\mathcal{P}(g.p)(X) = \alpha(g.g^{-1}Xg.p) = \text{Ad}_g^* \mathcal{P}(p)(X) \quad (10)$$

where the coadjoint action on an element ξ of \mathcal{G}^* is defined as:

$$\text{Ad}_g^* \xi(X) = \xi(g^{-1}Xg)$$

The stabilizer G_μ of $\mu \in \mathcal{G}^*$ acts on N_μ . The reduced phase space is precisely obtained by taking the quotient (assumed well-behaved):

$$\mathcal{F}_\mu = N_\mu / G_\mu \quad (11)$$

It is known that this is a symplectic manifold.

We then need to compute the Poisson bracket of functions on \mathcal{F}_μ . These functions are conveniently described as G_μ invariant functions on N_μ . To compute their Poisson bracket we first extend them arbitrarily in the vicinity of N_μ . Two extensions differ by a function vanishing on N_μ . The difference of the Hamiltonian vector fields of two such extensions is controlled by the following:

Lemma 3.1 *Let f be a function defined in a vicinity of N_μ and vanishing on N_μ . Then the Hamiltonian vector field X_f associated to f is tangent to the orbit $G.p$ at any point $p \in N_\mu$.*

As a consequence of this lemma we have a method to compute the reduced Poisson bracket. We take two functions defined on N_μ and invariant under G_μ and extend them arbitrarily. Then we compute their Hamiltonian vector fields on N and project them on the tangent space to N_μ by adding a vector tangent to the orbit $G.p$. These projections are independent of the extensions and the reduced Poisson bracket is given by the value of the symplectic form on N acting on them.

Proposition 3.1 *At each point $p \in N_\mu$ one can choose a vector $V_{f.p} \in \mathcal{G}.p$ such that $X_f + V_{f.p} \in T_p(N_\mu)$ and $V_{f.p}$ is determined up to a vector in $\mathcal{G}_\mu.p$.*

One finally gets the consistent general formula for reduced Poisson brackets:

Proposition 3.2 *The reduced Poisson bracket of two functions on \mathcal{F}_μ can be computed using any extensions f, g in the vicinity of N_μ according to:*

$$\{f, g\}_{\text{reduced}} = \{f, g\} + \frac{1}{2} ((V_{g.p}).f - (V_{f.p}).g) \quad (12)$$

This is equivalent to the Dirac bracket.

3.2 The case $N = T^*G$

Let now $M = G$ be a Lie group; one uses the left translations to identify $N = T^*G$ with $G \times \mathcal{G}^*$.

$$\omega \in T_g^*(G) \longrightarrow (g, \xi) \quad \text{where} \quad \omega = L_{g^{-1}}^* \xi \quad (13)$$

The Poisson structure on $N = T^*G$ is easily seen to be:

$$\{\xi(X), \xi(Y)\} = -\xi([X, Y]) ; \quad \{\xi(X), g\} = -g \cdot X ; \quad \{g, g\} = 0 \quad (14)$$

Geodesics on the group G correspond to left translations of 1-parameter groups (the tangent vector is transported parallel to itself), therefore $\frac{d}{dt}(g^{-1}\dot{g}) = 0$. This is a Hamiltonian system whose Hamiltonian is: $H = \frac{1}{2} (\xi, \xi)$. where we have identified \mathcal{G}^* and \mathcal{G} through the invariant Killing metric.

Here H is bi-invariant, so one can reduce this dynamical system using Lie subgroups H_L and H_R of G of Lie algebras \mathcal{H}_L and \mathcal{H}_R , acting respectively on the left and on the right on T^*G in order to obtain a non-trivial result.

Using the coordinates (g, ξ) on T^*G this action reads:

$$((h_L, h_R), (g, \xi)) \rightarrow (h_L g h_R^{-1}, \text{Ad}_{h_R}^* \xi)$$

We have written this action as a left action on T^*G , in order to apply the formalism developed in Section 3.1.

The moments are:

$$\mathcal{P}^L(g, \xi) = P_{\mathcal{H}^*} \text{Ad}_g^* \xi ; \quad \mathcal{P}^R(g, \xi) = -P_{\mathcal{H}^*} \xi ; \quad \mathcal{P} = (\mathcal{P}^L, \mathcal{P}^R) \quad (15)$$

where we have introduced the projector on \mathcal{H}^* of forms in \mathcal{G}^* induced by the restriction of these forms to \mathcal{H} .

3.3 The Calogero-Moser models

We follow here the derivation of [12]. Let us consider an involutive automorphism σ of a simple Lie group G and the subgroup H of its fixed points. Then H acts on the right on G defining a principal fiber bundle of total space G and base G/H , which is a global symmetric space. Moreover G acts on the left on G/H and in particular so does H itself. We shall consider the situation described in Section 3.2 when $H_L = H_R = H$. The Hamiltonian of the geodesic flow on G/H is invariant under the H action allowing to construct the

Hamiltonian reduction which under suitable choices of the momentum leads to the Calogero–Moser models. As a matter of fact since the phase space of the Calogero model is non-compact one has to start from a non-compact Lie group G and quotient it by a maximal compact subgroup H so that the symmetric space G/H is of the non-compact type. The derivative of σ at the unit element of G is an involutive automorphism of \mathcal{G} also denoted σ . Let us consider its eigenspaces \mathcal{H} and \mathcal{K} associated with the eigenvalues $+1$ and -1 respectively. Thus we have a decomposition:

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{K} \quad (16)$$

in which \mathcal{H} is the Lie algebra of H which acts by inner automorphisms on the vector space \mathcal{K} ($h\mathcal{K}h^{-1} = \mathcal{K}$).

Let \mathcal{A} be a maximal commuting set of elements of \mathcal{K} . It is called a Cartan algebra of the symmetric space G/H . It is known that every element in \mathcal{K} is conjugated to an element in \mathcal{A} by an element of H . Moreover \mathcal{A} can be extended to a maximal commutative subalgebra of \mathcal{G} by adding to it a suitably chosen abelian subalgebra \mathcal{B} of \mathcal{H} . We shall use the radical decomposition of \mathcal{G} under the abelian algebra \mathcal{A} :

$$\mathcal{G} = \mathcal{A} \oplus \mathcal{B} \bigoplus_{e_\alpha, \alpha \in \Phi} \mathbb{R}e_\alpha \quad (17)$$

These decompositions of \mathcal{G} exponentiate to similar decompositions of G . First $G = KH$ where $K = \exp(\mathcal{K})$. Then $A = \exp(\mathcal{A})$ is a maximal totally geodesic flat submanifold of G/H and any element of K can be written as $k = hQh^{-1}$ with $Q \in A$ and $h \in H$. It follows that any element of G can be written as $g = h_1Qh_2$ with $h_1, h_2 \in H$.

Of course this decomposition is non unique. This non-uniqueness is described in the following:

Proposition 3.3 *If $g = h_1Qh_2 = h'_1Q'h'_2$ we have: $h'_1 = h_1d^{-1}h_0^{-1}$, $h'_2 = h_0dh_2$ and $Q' = h_0Qh_0^{-1}$ where $d \in \exp(\mathcal{B}) = B$ and $h_0 \in H$ is a representative of an element of the Weyl group of the symmetric space. So if we fix $Q = \exp(q)$ such that q be in a fundamental Weyl chamber, the only ambiguity resides in the element $d \in B$.*

The reduction to Calogero-Moser models is then obtained by an adequate choice of the momentum $\mu = (\mu^L, \mu^R)$ such that $\mathcal{P} = \mu$. We take $\mu^R = 0$ so that the isotropy group of the right component is H_R itself.

The choice of the moment μ^L is of course of crucial importance. It must be fixed so that:

- its isotropy group H_μ is a maximal proper Lie subgroup of H , so that the phase space of the reduced system be of minimal dimension but non trivial.
- In order to ensure the unicity of the decomposition introduced in the Proposition (3.3) on N_μ we need:

$$\mathcal{H}_\mu \cap \mathcal{B} = \{0\} \quad (18)$$

We choose a complementary maximal isotropic subspace \mathcal{C} so that

$$\mathcal{H} = \mathcal{H}_\mu \oplus \mathcal{B} \oplus \mathcal{C} \quad (19)$$

and χ is a non-degenerate skew-symmetric bilinear form on $\mathcal{B} \oplus \mathcal{C}$, hence $\dim \mathcal{B} = \dim \mathcal{C}$. Notice that \mathcal{C} is defined up to a symplectic transformation preserving \mathcal{B} .

- The reduced phase space \mathcal{F}_μ has dimension $2 \dim \mathcal{A}$

We now construct a section \mathcal{S} of the bundle N_μ over \mathcal{F}_μ so that one can write:

$$N_\mu = H_\mu S H \quad (20)$$

To construct this section we take a point Q in A and an $L \in \mathcal{G}^*$ such that the point (Q, L) is in N_μ . In this subsection we shall for convenience identify \mathcal{G} and \mathcal{G}^* under the Killing form assuming that G is semi-simple. Moreover since the automorphism σ preserves the Killing form, \mathcal{H} and \mathcal{K} are orthogonal, and $P_{\mathcal{H}^*}$ reduces to the orthogonal projection on \mathcal{H} . Since $\mu^R = 0$ we have $L \in \mathcal{K}$ and one can write:

$$L = p + \sum_{e_\alpha, \alpha \in \Phi'} l_\alpha (e_\alpha - \sigma(e_\alpha)) \quad (21)$$

where $p \in \mathcal{A}$. From equation (15) one gets:

$$\mu^L = P_{\mathcal{H}} \left(p + \sum_{\alpha} l_\alpha (Q e_\alpha Q^{-1} - Q \sigma(e_\alpha) Q^{-1}) \right)$$

Since $Q = \exp(q)$, $q \in \mathcal{A}$ we have $Q e_\alpha Q^{-1} = \exp(\alpha(q)) e_\alpha$ and similarly $Q \sigma(e_\alpha) Q^{-1} = \exp(-\alpha(q)) \sigma(e_\alpha)$.

Then the above equation becomes:

$$\mu^L = \sum_{\alpha} l_\alpha \sinh \alpha(q) (e_\alpha + \sigma(e_\alpha)) \quad (22)$$

One can choose the momentum of the form: $\mu^L = \sum_{\alpha} g_\alpha (e_\alpha + \sigma(e_\alpha))$ namely μ^L has no component in \mathcal{B} , where the g_α are such that H_μ is of maximal dimension (we shall see that it essentially fixes them, and obviously if $g_\alpha \neq 0$ for any α equation (18) is automatically satisfied) and we have shown the:

Proposition 3.4 *The couples (Q, L) with $Q = \exp(q)$ and*

$$L = p + \sum_{\alpha} \frac{g_\alpha}{\sinh \alpha(q)} (e_\alpha - \sigma(e_\alpha))$$

with $p, q \in \mathcal{A}$ form a submanifold in N_μ of dimension $2 \dim \mathcal{A}$.

Notice that L is just the Lax operator of the Calogero model and that the section \mathcal{S} depends of $2 \dim \mathcal{A}$ parameters in an immersive way. Hence one can identify N_μ with the set of orbits of \mathcal{S} under $H_\mu \times H$ i.e. the set of points $(g = h_1 Q h_2, \xi = h_2^{-1} L h_2)$ with $h_1 \in H_\mu$ and $h_2 \in H$ uniquely defined due to condition (18). The variables p and q appearing in Q and L are the dynamical variables of the Calogero model and form a pair of canonically conjugate variables.

We then compute the Poisson bracket of the functions on \mathcal{F}_μ whose expressions on the section \mathcal{S} are $L(X)$ and $L(Y)$ for $X, Y \in \mathcal{K}$. These functions have uniquely defined $H_\mu \times H$ invariant extensions to N_μ given respectively by:

$$F_X(g, \xi) = \langle \xi, h_2^{-1} X h_2 \rangle, \quad F_Y(g, \xi) = \langle \xi, h_2^{-1} Y h_2 \rangle \quad \text{where } g = h_1 Q h_2$$

Notice that h_2 is a well-defined function of g in N_μ due to condition (18). According to the prescription given in the section 3.1 we choose extensions of these functions in the vicinity

of N_μ . We *define* these extensions at the point $p = (g, \xi) \in T^*G$ by the *same* formulae in which h_2 is chosen to be a function depending *only* on g and reducing to the above-defined h_2 when $p \in N_\mu$. Because of the non-uniqueness of the decomposition $g = h_1 Q h_2$ outside N_μ one cannot assert that the functions F_X, F_Y are invariant under the action of $H \times H$ and we must appeal to the general procedure to compute the reduced Poisson brackets.

The complete derivation with all its technical subtleties can be found in [18]. The final result gives the general r -matrix for trigonometric Calogero-Moser models in so-called dual form [30]:

Theorem 3.1 *There exists a linear mapping $R: \mathcal{K} \rightarrow \mathcal{H}$ such that:*

$$\{L(X), L(Y)\}_{\text{reduced}} = L([X, RY] + [RX, Y]) \quad (23)$$

and R is given by:

$$R(X) = \nabla_g h_2(X) + \frac{1}{2} D_Q(V_X) \quad (24)$$

where:

Proposition 3.5 *On the section \mathcal{S} with $Q = \exp(q) \in A$ we have:*

For $X \in \mathcal{K}$ i.e. $X = X_0 + \sum X_\alpha(e_\alpha - \sigma e_\alpha)$, $X_0 \in \mathcal{A}$

$$\nabla_g h_2(X) = -h_0(X) + \sum_\alpha X_\alpha \coth(\alpha(q))(e_\alpha + \sigma e_\alpha) \quad (25)$$

Here $h_0(X)$ is a linear function from \mathcal{G} to \mathcal{B} which is fixed by the condition:

$$X_L \equiv h_0(X) - \sum_\alpha \frac{X_\alpha}{\sinh \alpha(q)} (e_\alpha + \sigma e_\alpha) \in \mathcal{H}_\mu \oplus \mathcal{C} \quad (26)$$

3.4 Two examples

To illustrate the power of this method we now give two examples of r -matrices. The A_n case had already been treated in [17] and serves as a check on the validity of the derivation. The case of $SU(n, n)$ Calogero-Moser model proved to be too intricate for a direct computation; however this method immediately gives its r -matrix.

The standard Calogero-Moser model ($SL(n)$) is obtained by starting from the non compact group $G = SL(n, \mathbb{C})$ and its maximal compact subgroup $H = SU(n)$ as first shown by [12]. We choose the momentum μ_L as described in Section (3.1) so that the isotropy group H_μ be a maximal proper Lie subgroup of H . Obviously one can take μ_L of the form:

$$\mu_L = i(vv^+ - 1) \quad (27)$$

where v is a vector in \mathbb{C}^n such that $v^+v = 1$, hence μ_L is a traceless antihermitian matrix. Then $g\mu_L g^{-1} = \mu_L$ if and only if $gv = cv$ where c is a complex number of modulus 1. Hence $H_\mu = S(U(n-1) \times U(1))$ which has the above-stated property.

In this case the automorphism σ is given by $\sigma(g) = (g^+)^{-1}$ (notice that we consider only the real Lie group structure), B is the group of diagonal matrices of determinant 1 with pure phases on the diagonal and A is the group of real diagonal matrices with determinant 1. The property (18) is then satisfied as soon as the vector v has no zero component. As a

matter of fact, v is further constrained by μ_L being a value of the moment map. Considering equation (22) we see that μ_L has no diagonal element, which implies that all the components of v are pure phases $v_j = \exp(i\theta_j)$. These extra phases which will appear in the Lax matrix can however be conjugated out by the adjoint action of a constant matrix $\text{diag}(\exp(i\theta_j))$ hence we shall from now on set $v_j = 1$ for all j . This is the solution first considered by Olshanetskii and Perelomov.

The Lax matrix L is then given by Proposition (3.4) and therefore

$$L = p + \sum_{k < l} \frac{1}{\sinh(q_k - q_l)} (iE_{kl} - iE_{lk})$$

The r -matrix can now be deduced straightforwardly from Proposition (3.5), after re-converting the dual form where R is an endomorphism of the Lie algebra into the more usual direct form where R lives in the tensor product of the Lie algebra by itself. One ends up with:

$$R_{12} = \sum_{k \neq l} \coth(q_k - q_l) E_{kl} \otimes E_{lk} + \frac{1}{2} \sum_{k \neq l} \frac{1}{\sinh(q_k - q_l)} (E_{kk} - \frac{1}{n} \mathbf{1}) \otimes (E_{kl} - E_{lk})$$

This gives back the already known r -matrix of the Calogero model for the potential $1/\sinh(x)$, and the other potentials $1/\sin(x)$ and $1/x$ have similar r -matrices obtained by analytic continuation.

The $SU(n, n)$ Calogero model is obtained by starting from the non-compact group $G = SU(n, n)$. This is the subgroup of $SL(2n, \mathbb{C})$ which leaves invariant the sesquilinear quadratic form defined by

$$Q((u_1, v_1), (u_2, v_2)) = \begin{pmatrix} u_1^+ & v_1^+ \end{pmatrix} J \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = u_1^+ v_2 + v_1^+ u_2 \quad (28)$$

where u_i, v_i are vectors in \mathbb{C}^n and J is the matrix

$$J = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}.$$

The Lie algebra of $SU(n, n)$ therefore consists of block matrices

$$\mathcal{G} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a = -d^+, \text{Tr}(a + d) = 0, b^+ = -b, c^+ = -c \right\} \quad (29)$$

where a, b, c, d are $n \times n$ complex matrices.

We consider again the automorphism $\sigma : \sigma(g) = (g^+)^{-1}$, which can be consistently restricted to $SU(n, n)$. Its fixed points at the Lie algebra level consist of block matrices

$$\mathcal{H} = \left\{ \begin{pmatrix} a & c \\ c & a \end{pmatrix} \mid a^+ = -a, \text{Tr}(a) = 0, c^+ = -c \right\} \quad (30)$$

This Lie algebra is isomorphic to the Lie algebra of $S(U(n) \times U(n))$, the two $u(n)$'s being realized respectively by $a + c$ and $a - c$.

The subalgebra \mathcal{B} consists of matrices of the form (30) with $c = 0$, and a is a diagonal matrix of zero trace and purely imaginary coefficients. The Abelian subalgebra \mathcal{A} consists of matrices of the form (29) with $b = c = 0$ and $a = -d$ being a real diagonal matrix.

To perform the reduction, we choose as above $\mu^R = 0$ and

$$\mu^L = i(vv^+ - \mathbf{1}) + i\gamma J \quad (31)$$

The vector v has again $2n$ components all equal to 1.

Notice that in equation (31) the parameter γ is an arbitrary real number. This will lead to existence of a second coupling constant in the corresponding Calogero model.

Then, from proposition (3.4), the Lax matrix is found to be:

$$\begin{aligned} L = & p + \sum_{i < j} \frac{1}{\sinh(q_i - q_j)} (1 - \sigma)(iE_{ij} + iE_{j+n, i+n}) \\ & + \sum_{i < j} \frac{1}{\sinh(q_i + q_j)} (1 - \sigma)(iE_{i, j+n} + iE_{j, i+n}) \\ & + (\gamma + 1) \sum_i \frac{1}{\sinh(2q_i)} (1 - \sigma)(iE_{i, i+n}) \end{aligned}$$

where p is a generic element of \mathcal{A} of the form $\text{diag } p_i, -\text{diag } p_i$.

The r -matrix is then computed straightforwardly:

$$\begin{aligned} R_{12} = & \frac{1}{2} \sum_{k \neq l} \coth(q_k - q_l) (E_{kl} + E_{k+n, l+n}) \otimes (E_{lk} - E_{l+n, k+n}) \\ & + \frac{1}{2} \sum_{k, l} \coth(q_k + q_l) (E_{k, l+n} + E_{k+n, l}) \otimes (E_{l+n, k} - E_{l, k+n}) \\ & + \frac{1}{2} \sum_{k \neq l} \frac{1}{\sinh(q_k - q_l)} (E_{kk} + E_{k+n, k+n} - \frac{1}{n} \mathbf{1}) \otimes (E_{kl} - E_{k+n, l+n}) \\ & + \frac{1}{2} \sum_{k, l} \frac{1}{\sinh(q_k + q_l)} (E_{kk} + E_{k+n, k+n} - \frac{1}{n} \mathbf{1}) \otimes (E_{k, l+n} - E_{k+n, l}) \end{aligned}$$

These dynamical r -matrices depend only on the dynamical variable q . The different approach advocated in [24] leads to r -matrices depending on both p and q variables, but on a smaller set of algebra generators.

4 The dynamical r matrices of Calogero and Ruijsenaars models

Dynamical r -matrices have been derived for the Calogero-Moser and (relativistic) Ruijsenaars Schneider models using either the technique described here or a direct method starting from an ansatz of the same form. We will now describe the results achieved in this way for A_n models, and indicate interesting and sometimes deep connections between these various r -matrices.

Let us start with **Calogero-Moser models**. The rational and trigonometric matrices were described in the previous section.. The elliptic case was solved by Sklyanin and by

Braden et al. [17]. The Lax matrix reads:

$$L(\lambda) = \sum_{i=1}^N p_i e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^N l(q_{ij}, \lambda) e_{ij} \quad (32)$$

Here one has set:

$$l(x, \lambda) = -\frac{\sigma(x + \lambda)}{\sigma(x) \sigma(\lambda)}, \quad V(x) = \wp(x) \quad (33)$$

where σ and \wp are Weierstrass elliptic functions. The classical r -matrix reads:

$$\begin{aligned} r_{12}(\lambda, \mu) &= \sum_{\substack{i,j=1 \\ i \neq j}}^N l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N l(q_{ij}, \mu) (e_{ii} + e_{jj}) \otimes e_{ij} \\ &\quad - [\zeta(\lambda - \mu) + \zeta(\mu)] \sum_{i=1}^N e_{ii} \otimes e_{ii}. \end{aligned} \quad (34)$$

Note that a spectral parameter is now present in L and r . This particular formulation of the spinless elliptic case is due to Krichever [15]. The other known Lax formulation due to Olshanetskii and Perelomov [12] has no spectral parameter but requires a p and q dependence in the r -matrix, which was only recently given in [24] and has a totally different algebraic form.

The spin Calogero-Moser models were introduced in [9]. The Lax operator for the elliptic case reads:

$$L(\lambda) = \sum_{i=1}^N p_i e_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^N l(q_{ij}, \lambda) f_{ij} e_{ij} \quad (35)$$

where f_{ij} are spinlike variables with the Kirillov-Poisson bracket structure:

$$\{f_{ij}, f_{kl}\} = \frac{1}{2} (\delta_{il} f_{jk} + \delta_{ki} f_{lj} + \delta_{jk} f_{il} + \delta_{lj} f_{ki}). \quad (36)$$

One then needs to introduce a parametrization of f_{ij} so as to be on a coadjoint orbit of $SU(N)$: Introducing vectors

$$\begin{aligned} (\xi_i)_{i=1 \dots N} &\quad \text{with} \quad \xi_i = (\xi_i^a)_{a=1 \dots r} \\ (\eta_i)_{i=1 \dots N} &\quad \text{with} \quad \eta_i = (\eta_i^a)_{a=1 \dots r} \end{aligned}$$

with the Poisson brackets

$$\{\xi_i^a, \xi_j^b\} = 0, \quad \{\eta_i^a, \eta_j^b\} = 0, \quad \{\xi_i^a, \eta_j^b\} = -\delta_{ij} \delta_{ab}, \quad (37)$$

we parametrize f_{ij} as follows:

$$f_{ij} = \langle \xi_i | \eta_j \rangle = \sum_{a=1}^r \xi_i^a \eta_j^a. \quad (38)$$

The phase space now becomes a true symplectic manifold.

The Hamiltonian takes the form

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 - \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N f_{ij} f_{ji} V(q_{ij}), \quad q_{ij} = q_i - q_j \quad (39)$$

with the Weierstrass function as elliptic potential, as in the spinless case.

The classical r -matrix then reads:

$$\begin{aligned} r_{12}(\lambda, \mu) &= \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N l(q_{ij}, \lambda - \mu) e_{ij} \otimes e_{ji} + \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^N l(q_{ij}, \lambda + \mu) e_{ij} \otimes e_{ij} \\ &- \frac{1}{2} [\zeta(\lambda + \mu) + \zeta(\lambda - \mu)] \sum_{i=1}^N e_{ii} \otimes e_{ii}. \end{aligned} \quad (40)$$

Trigonometric and rational cases can be derived from the elliptic case by taking suitable limits [19]. The spinless case can also be derived from the spin case by taking $r = 1$ and introducing a further Hamiltonian reduction by the action of $U(1)$ as a phase on the vectors ξ_i, η_i . The supplementary terms in the spinless r matrix arise from the conjugation of the Lax matrix required to bring it in canonical shape (32) after elimination of the vector-like degrees of freedom. Let us finally remark that this r -matrix structure has yielded a number of important developments: exact classical Yangian symmetry [19] (a quantum version of it had been found beforehand, using heavy direct algebraic computations [37]); quantum version of the dynamical r -matrix using the shifted version of the quantum Yang-Baxter equation described in [39, 38, 40].

The spinless relativistic RS models are described by the hamiltonian:

$$H = mc^2 \sum_{j=1}^N (\cosh \theta_j) \prod_{k \neq j} f(q_k - q_j) \quad (41)$$

where

$$\begin{aligned} f(q) &= \left(1 + \frac{g^2}{q^2}\right)^{1/2} \quad (\text{rational}) \\ f(q) &= \left(1 + \frac{\alpha^2}{\sinh^2 \frac{\nu q}{2}}\right)^{1/2} \quad (\text{hyperbolic}) \\ f(q) &= (\lambda + \nu \mathcal{P}(q)) \quad (\text{elliptic}), \quad \mathcal{P} = \text{Weierstrass function} \end{aligned} \quad (42)$$

Here the canonical variables are a set of rapidities $\{\theta_i, i = 1 \cdots N\}$ and conjugate positions q_i such that $\{\theta_i, q_j\} = \delta_{ij}$.

The dynamical system admits a Lax representation with the Lax operator:

$$\begin{aligned} L &= \sum_{j,k=1}^N L_{jk} e_{jk} \\ L_{jk} &= \exp \beta \theta_j \cdot C_{jk} (q_j - q_k) \cdot \left(\prod_{m \neq j} f(q_j - q_m) \prod_{l \neq k} f(q_l - q_k) \right)^{1/2} \end{aligned} \quad (43)$$

where $\{e_{jk}\}$ is the usual basis for $N \times N$ matrices; f was given in (42) and

$$\begin{aligned} C_{jk}(q) &= \frac{\gamma}{\gamma + iq} \quad (\text{rational}) \\ C_{jk}(q) &= \left(\cosh \frac{\nu}{2} q + ia \sinh \frac{\nu}{2} q \right)^{-1} \quad (\text{trigonometric}) \end{aligned} \quad (44)$$

$$C_{jk}(q) = \frac{\Phi(q + \gamma, \lambda)}{\Phi(\gamma, \lambda)} \quad (\text{elliptic}) \quad (45)$$

Again the function Φ is defined as:

$$\Phi(x, \lambda) \equiv \frac{\sigma(x + \lambda)}{\sigma(x)\sigma(\lambda)} \quad (46)$$

where σ is the Weierstrass function.

The elliptic r -matrix structure is better written as a quadratic expression in terms of the Lax operator [22]:

$$\begin{aligned} \{L_1(\lambda), L_2(\mu)\} &= (L_1(\lambda) \otimes L_2(\mu))a_1(\lambda, \mu) - a_2(\lambda, \mu)(L_1(\lambda) \otimes L_2(\mu)) \\ &\quad + (\mathbf{1} \otimes L_2(\mu))s_1(\lambda, \mu)(L_1(\lambda) \otimes \mathbf{1}) \\ &\quad - (L_1(\lambda) \otimes \mathbf{1})s_2(\lambda, \mu)(\mathbf{1} \otimes L_2(\mu)) \end{aligned} \quad (47)$$

Here one defines:

$$\begin{aligned} a_1(\lambda, \mu) &= a(\lambda, \mu) + w, \quad s_1(\lambda, \mu) = s(\lambda, \mu) - w \\ a_2(\lambda, \mu) &= a(\lambda, \mu) + s(\lambda) - s^*(\mu) - w, \quad s_2(\lambda, \mu) = s^*(\mu) + w \end{aligned} \quad (48)$$

The matrices a and s are obtained from the r -matrix of the elliptic Calogero-Moser model given in (34) as $r(\lambda, \mu) \equiv a(\lambda, \mu) + s(\lambda)$ where a is the skew-symmetric matrix:

$$a(\lambda, \mu) = -\zeta(\lambda - \mu) \sum_{k=1}^N E_{kk} \otimes E_{kk} - \sum_{k \neq j} \Phi(q_j - q_k, \lambda - \mu) E_{jk} \otimes E_{kj} \quad (49)$$

and s, s^* are non-skew-symmetric matrices independent of the second spectral parameter:

$$\begin{aligned} s(\lambda, \mu) &= \zeta(\lambda) \sum_{k=1}^N E_{kk} \otimes E_{kk} + \sum_{k \neq j} \Phi(q_j - q_k, \lambda) E_{jk} \otimes E_{kk} \\ s^*(\lambda, \mu) &= \zeta(\lambda) \sum_{k=1}^N E_{kk} \otimes E_{kk} + \sum_{k \neq j} \Phi(q_j - q_k, \lambda) E_{kk} \otimes E_{jk} \end{aligned} \quad (50)$$

and finally w is a supplementary matrix, independent of the spectral parameters:

$$w = \sum_{k \neq j} \zeta(q_k - q_j) E_{kk} \otimes E_{jj} \quad (51)$$

This r -matrix structure is a Sklyanin-type bracket (although realized in the more generic case of an initially dynamical r -matrix) obtained from the Calogero-Moser r -matrix structure

viewed as a linear bracket. This can be interpreted from the fact that RS models are obtained not only as hamiltonian reductions from current algebras on elliptic curves but also alternatively [13] as hamiltonian reductions from Heisenberg double of Lie groups [41]. In this case the initial Poisson structure on the large phase space is itself a quadratic bracket instead of the canonical initial linear (Kirillov) bracket which is the natural structure on the cotangent bundle of a Lie group. This relation is maintained throughout the hamiltonian reduction procedure and the final r -matrix structures are essentially connected in the same way.

The previously obtained r -matrices [21] can be obtained from this one by sending one period of the elliptic functions to infinity and suitably conjugating the Lax pair in such a way as to get a completely symmetric expression in terms of the momenta θ_i . On the other hand the r -matrix found in [20] cannot be easily inserted in this scheme. In fact it correspond to a very specific value of the parameters where the Lax matrix becomes completely symmetric and the r -matrix may only then take this very special form.

The classical r -matrices admit a quantization scheme on the same lines as the Calogero-Moser case [7]. It was developed in [42].

Finally a word about **the spin RS dynamical system**. They were introduced in [11]. It is not clear at this time how to define a consistent hamiltonian structure in the most general case although the rational case was solved recently [23]. The task is indeed easier here since there exists a duality symmetry [43] connecting the rational RS model to the trigonometric CM model for which the spin model is well known. Let us finally mention that there exists a general scheme to obtain Hamiltonian structures from Lax representations using the tools of algebraic geometry [44] (see also Pr. Krichever's contribution to this colloquium) and this scheme now appears to be the most promising way to get these elusive Hamiltonian structures.

Acknowledgements

The works presented here were done in collaboration with O. Babelon, E. Billey, G. Rollet and M. Talon at the LPTHE Paris VI (URA CNRS 280). I have also drawn upon works by H.W. Braden, Yu. B. Suris and E.K. Sklyanin. I wish to thank A. Antonov, H.W. Braden and J.M. Maillet for clarifications of particular points in this presentation; S.N.M. Ruijsenaars for suggesting to write a summary of the current situation as Section 4; and J.F. Van Diejen and L. Vinet for their kind invitation to attend this workshop.

References

- [1] A.V. Tsiganov, *Automorphisms of $sl(2)$ and dynamical r -matrices*. solv-int 9610003.
- [2] Y.B. Zeng, J. Hietarinta, “*Classical Poisson structures and r matrices from constrained flows*”, Journ. Phys. **A**: (Math. Gen.) **29** (1996), 5241, solv-int 9509005.
- [3] P.P. Kulish, S. Rauch-Wojciechowski, A.V. Tsiganov, *Stationary problems for equation of the KdV type and dynamical r -matrices*, Mod. Phys. Lett. **A9** (1994), 2063.
- [4] A.A. Belavin, V.G. Drin’feld, *Triangle equations and simple Lie algebras*, Sov. Scientific Rev. **C 4** (1984), 93.
- [5] O. Schiffmann, *On classification of dynamical r -matrices*, q-alg 9706017.
- [6] G. Felder, C. Wierczkowski, *Conformal blocks on elliptic curves and the Knizhnik-Zamolodchikov-Bernard equations*, Comm. Math. Phys. **176** (1996), 133.
- [7] , J.Avan, O. Babelon, E. Billey, “*The Gervais Neveu Felder equation and the quantum Calogero Moser systems*”, Comm. Math. Phys. **178** (1996), 50, hep-th 9505091.
- [8] F. Calogero, *Exactly solvable one-dimensional many-body problems*, Lett. Nuovo Cimento **13** (1975), 411. *On a functional equation connected with integrable many-body problems*, Lett. Nuovo Cimento **16** (1976), 77.
J. Moser, *Three integrable Hamiltonian systems connected to isospectral deformations*, Adv. Math. **16** (1976), 1.
- [9] J. Gibbons, T. Hermesen, *A generalisation of the Calogero-Moser system*, Physica **11D** (1984), 337.
- [10] S.N.M. Ruijsenaars, *Complete integrability of relativistic Calogero-Moser systems and elliptic function identities*, Commun. Math. Phys. **110** (1987), 191.
J.F. van Deijen, *Families of commuting difference operators*, Ph.D. Thesis, University of Amsterdam (1994).
- [11] I. Krichever, A. Zabrodin, *Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra*, hep-th/9505039.
- [12] M.A. Olshanetsky, A.M. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Rep. **71** (1981), 314. *Quantum integrable systems related to Lie algebras*, Phys. Rep. **94** (1983), 313.
- [13] G.E. Arutyunov, S.A.Frolov, P.B. Medvedev, *Elliptic Ruijsenaars Schneider model via the Poisson reduction of the affine Heisenberg double*, hep-th 9607170.
- [14] A. Gorsky, N. Nekrasov, *Elliptic Calogero-Moser system from two-dimensional current algebra*, Nucl. Phys. **B436** (1995), 332, hep-th/9401021. *Relativistic Calogero-Moser model as gauged WZW theory*, Theor. Math. Phys. **100** (1994), 874, hep-th/9401017.
A. Gorsky, *Integrable many body systems in the field theories*, Theor. Math. Phys. **103** (1995), 681, hep-th/9410228.

- [15] I. Krichever, *Elliptic solutions of the Kadomtsev- Petviashvili equation and integrable systems of particles*, Func. Anal. Appl. **14** (1980), 282.
- [16] I.M. Krichever, O. Babelon, E. Billey, M. Talon, *Spin generalization of the Calogero Moser system and the matrix KP equation*, Amer. Math. Soc. Trans. **170** (1995), 83, hep-th 9411160.
- [17] J. Avan, M. Talon, *Classical r -matrix structure for the Calogero model*, Phys. Lett. **B 303** (1993), 33.
E.K. Sklyanin, *Dynamical r -matrices for the elliptic Calogero-Moser model*, Alg. and Anal. **6** (1994), 227, hep-th/9308060.
H.W. Braden, T. Suzuki, *r -matrices for elliptic Calogero-Moser models*, Lett. Math. Phys. **30** (1994), 147, hep-th/9309033.
- [18] J. Avan, O. Babelon, M. Talon, *Construction of the classical r -matrices for the Toda and Calogero models*, Alg. and Anal. **6** (1995), 255.
- [19] E. Billey, J. Avan, O. Babelon, *The r -matrix structure of the Euler-Calogero-Moser model*, Phys. Lett. **A 186** (1994), 114, hep-th/9312042. *Exact Yangian symmetry in the classical Euler-Calogero-Moser model*, Phys. Lett. **A 188** (1994), 263, hep-th/9401117.
- [20] O. Babelon, D. Bernard, *The sine-Gordon solitons as an N -body problem*, Phys. Lett. **B 317** (1993), 363.
- [21] J. Avan, G. Rollet, *The classical r -matrix for the relativistic Ruijsenaars-Schneider system*, Phys. Lett. **A 212** (1996), 50, hep-th/9510166.
- [22] Yu. B. Suris, *Why is the Ruijsenaars-Schneider hierarchy...*, Phys. Lett. **A225** (1997), 253.
- [23] G.E. Arutyunov. S.A. Frolov, *On the Hamiltonian structure of the spin Ruijsenaars Schneider system*, hep-th 9703119.
- [24] H.W. Braden, *R -matrices and generalized inverses*, q-alg 9706001.
- [25] K. Hasegawa, *Ruijsenaars commuting difference operators as commuting transfer matrices*, q-alg 9512029. A. Antonov, K. Hasegawa, A. Zabrodin, *On trigonometric intertwining vectors and the non-dynamical r -matrix for the Ruijsenaars model*, hep-th 9704074.
- [26] L.D. Faddeev, L.A. Takhtadzyan, *Hamiltonian methods in the Theory of Solitons*, Springer Series in Soviet Mathematics (1987).
- [27] J. Liouville, Journ. Mat. (Liouville) **20** (1855), 137.
- [28] V.I. Arnol'd, *Mathematical Methods in Classical mechanics*, Graduate Text in Mathematics **60**, Springer Verlag (1978)
- [29] P.D. Lax, *Integrals of non linear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21**, (1968), 467.

- [30] E. K. Sklyanin, *On complete integrability of the Landau Lifschitz equation*, Zap. Nauch. Sem. LOMI **95** (1980), 55; M.A. Semenov-Tjan-Shanskii, *What is a classical R matrix ?*, Funct. Anal. Appl. **17** (1983), 17.
- [31] J.M. Maillet, *Kac Moody algebra and extended Yang Baxter relations in the $O(n)$ non linear sigma model* Phys. Lett **B 162** (1985), 137.
- [32] O. Babelon, C.M. Viallet, *Hamiltonian structures and Lax equations*, Phys. Lett **B 237** (1989), 411.
- [33] A.G. Reiman, M.A. Semenov-Tjan-Shanskii, *Group-theoretical methods in the theory of finite-dimensional integrable systems*, Encyclopedia of Mathematical Science **16**, pp 190-377, Springer Verlag (1990). J. Avan, M.Talon, *Graded r-matrices for integrable systems*, Nucl. Phys. **B352** (1991), 215.
- [34] E.K. Sklyanin, *Quantum version of the inverse scattering problem method*, J. Sov. Math. **19** (1981), 1546.
- [35] F. Magri, *A simple model of the integrable Hamiltonian equations*, J. Math. Phys. **19** (1978), 1156 ; I.M. Gel'fand, I. Dorfman, *Hamiltonian operators and algebraic structures related to them*, Funct. Anal. Appl. **13** (1979), 248; L.C. Li, S. Parmentier, *Nonlinear Poisson structures and r matrices*, Comm. Math. Phys. **125** (1989), 545.
- [36] R. Abraham, J.E. Marsden, *Foundations of mechanics*, Benjamin-Cummings (1978).
- [37] D. Bernard, M. Gaudin, F.D.M. Haldane, V. Pasquier, *Yang-Baxter equation in spin chains with long range interactions*, J. Phys. A: Math. Gen. **26** (1993), 5219, hep-th/9301084.
- [38] J.L. Gervais, A. Neveu, *Novel triangle relations and the absence of tachyons in Liouville string field theory*, Nucl. Phys. **B 238** (1984), 325.
- [39] J.M. Maillet, F.W. Nijhoff, *Integrability for multidimensional lattice models*, Phys. Lett. **B 224** (1989), 389.
- [40] G. Felder, *Conformal field theory and integrable systems associated to elliptic curves*, Proceedings of the ICM 1994, pp1247-1255, Birkhauser (1994), hep-th/9407154.
- [41] A. Yu. Alekseev, A.Z. Malkin, *Symplectic structures associated to Lie-Poisson groups*, Comm. Math. Phys **162** (1994), 147.
- [42] G.E. Arutyunov, S.A. Frolov, *Quantum dynamical R-matrices and quantum Frobenius group*, q-alg 9610009; G.E. Arutyunov, L.O. Chekhov, S.A. Frolov, *R-matrix quantization of the elliptic Ruijsenaars-Schneider model*, q-alg 9612032.
- [43] S.N.M. Ruijsenaars, *Action-angle maps and scattering theory for some finite-dimensional integrable systems*, Comm. Math. Phys. **115** (1988), 127.
- [44] I.M. Krichever, D.H. Phong, *On the integrable geometry of soliton equations and $N=2$ supersymmetric gauge theories*, hep-th 9604199.